Mathematics II

(English course)

Second semester, 2012/2013

Self-test

1. Consider the quadratic form

$$Q(x, yz) = x^2 + 2xy + y^2 + 4yz + 5z^2$$

   (a) Find the symmetric matrix A such that

$$Q(x, y, z) = (x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \forall (x, yz) \in \mathbb{R}^3.$$

   (b) Classify the quadratic form Q.

   (c) Write the characteristic polynomial for the matrix A.

2. Consider the function $f(x, y) = \frac{\sqrt{5-e^{x^2+y^2}}}{1-\ln(x^2-y^2)}$.

   (a) Find the domain of f and sketch it.

   (b) The domain of f is open? Why?

3. Find the value of $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2-x^2y}{x^2+y^2}$, or show that this limit does not exist.

4. Consider the function $f(x, y) = \frac{\sqrt{1+(x\ln y)^2}}{x^2+e^{y-1}}$.

   (a) Find $\nabla f(x, y)$.

   (b) Compute the derivative of f along the vector $(1, -2)$ at the point $(3, 1)$.

5. Consider the function

$$f(x, y) = \begin{cases} (x^2 - y^2) \ln \left(x^2 + y^2\right), & \text{for } (x, y) \neq (0, 0), \\ 0, & \text{for } (x, y) = (0, 0). \end{cases}$$

   (a) Find the function $\frac{\partial f}{\partial x}$.

   (b) Show that f is differentiable at the point $(0, 0)$.

(c) f is continuous? Why?

(d) Find the set of points where $\frac{\partial f}{\partial x}$ is continuous.

6. Assuming the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is homogeneous of degree -3 and $f(1,3) = \ln 2$, and $\frac{\partial f}{\partial y}(1,3) = -1$, find $\frac{\partial f}{\partial x}(1,3)$.

7. Find the critical points of the function $f(x,y) = e^{xy}(x^2 + y^2)$, and classify them.

8. Find the maximizers and minimizers of the function $f(x,y) = x^2 + y^2$ on the domain $D = \left\{(x,y) : (x+y+2)^2 + \frac{(x-y)^2}{4} \leq 1\right\}$.

9. Compute the following integrals

(a) $\int_A xy^2 dxdy$, with $A = \left\{(x,y) : \frac{x^2}{2} \leq y \leq x\right\}$.

(b) $\int_A \sqrt{1 - x^2 - y^2} dxdy$, with $A = \{(x,y) : (x^2+y^2)^2 \leq x^2 - y^2 \leq x\}$.

10. Solve the following Cauchy problems:

(a) $y - xy' = 1 + x^2 y'$ with $y(1) = 2$,

(b) $y'' + 2y' + 2y = e^x \sin x$, with $y(0) = 0$, $y'(0) = -2$.